SPECIAL RELATIVITY

1. Introduction

By the end of the nineteenth century, many physicists felt they had a pretty good understanding of the world around them, thanks to the two major fundamental theories of classical physics: Newton's laws of mechanics and gravitation, and Maxwell's laws of electromagnetism. For instance, the American physicist Albert Michelson (1852–1931) wrote in 1894, "The more important fundamental laws and facts of physical science have all been discovered, and these are now so firmly established that the possibility of their ever being supplanted in consequence of new discoveries is exceedingly remote . . . Our future discoveries must be looked for in the sixth place of decimals."

However, some physicists of the era, such as the British scientist Lord Kelvin (1824–1907), worried that there were still "two clouds" in the way of a perfectly clear theory. One cloud, which concerned the mysteries of black body radiation, would eventually lead to the theory of quantum mechanics. The other cloud concerned the inability to find evidence of the "aether" that the electromagnetic radiation of Maxwell's theory (including light waves) was supposed to propagate in, leading to an apparent conflict between Newton's laws and Maxwell's laws. To resolve this conflict, the Dutch physicist Hendrik Lorentz (1853–1928), among others, proposed that moving bodies experience time and space in a different way from stationary bodies, in particular experiencing *time dilation* and *length contraction*. These hypotheses were then explained by the German-born physicist Albert Einstein (1879–1955) in his theory of special relativity (which also abolished the need for an aether). This theory was then used to make many further predictions, the most famous of which was Einstein's equation $E = mc^2$ of *mass-energy equivalence*, which among other things suggested the presence of significant amounts of energy that could be released from nuclear reactions.

A thorough description of the special theory of relativity would require several lectures and a significant amount of background in classical physics. In this handout, we will only focus on a small slice of this theory, namely the theory of *Doppler shifts* (both in classical physics and in special relativity), and Einstein's famous equation $E = mc^2$. We will also focus more on the mathematical derivation than the physical consequences. For further reading in special relativity, I recommend the text "Spacetime physics" by Taylor and Wheeler (W.H. Freeman & Co., Second edition, 1992).

The special theory of relativity was later extended by Einstein to the *general theory of relativity*, which also takes into account the effect of gravity on distorting both space and time (and which is needed, among other things, in order to correctly calibrate the readings from GPS satellites that allow for devices such as smartphones to accurately determine location). However, we will not discuss this more complicated theory here.

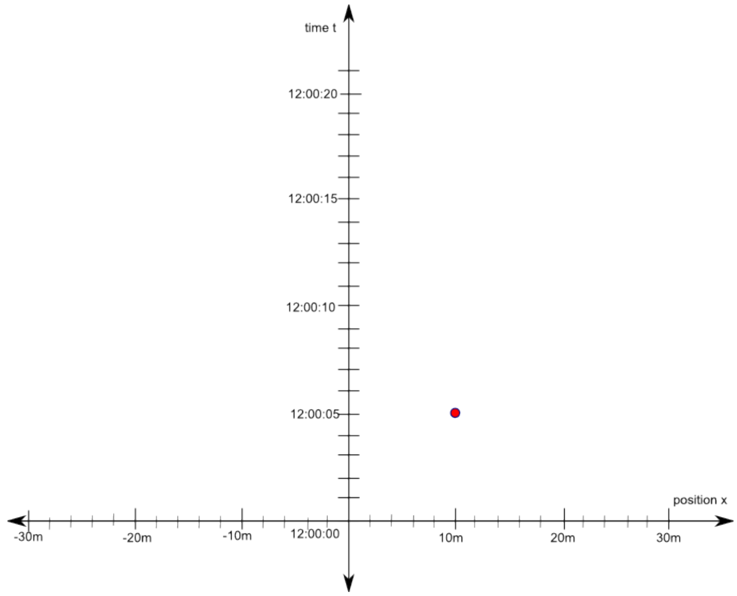

FIGURE 1. Spacetime. The dot indicates a location ten meters to the right of a fixed origin, five seconds after noon.

2. Spacetime diagrams

One of the basic philosophical insights in Einstein's theory of relativity is that the three dimensions of physical space and the one dimension of time should not be viewed as completely independent aspects of physical reality, but should instead be viewed as the two components of a unified four-dimensional *spacetime* on which all other physical objects reside. In the words of the mathematician Hermann Minkowski (1864-1909): "Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality.".

From the perspective of higher mathematics, four-dimensional space is not that much more complicated to deal with than three-dimensional space; one simply has to replace three-dimensional vectors (x, y, z) by four-dimensional ones (x, y, z, t). But when trying to *visualise* the geometry of space, four dimensions is of course much more difficult to deal with than three. So we will focus on a simplified model of spacetime, in which we only consider *one* dimension of space, together with one dimension of time, leading to a spacetime that is only *two* dimensional. This allows for spacetime to be displayed on a page or screen in what is known as a *spacetime diagram*. It looks very similar to the familiar Cartesian plots used to graph functions, except that instead of having x and y axes, we have an x axis (representing space) and a t axis (representing time): see Figure 1.

A physical object (such as a human being) does not occupy just one point in spacetime; instead, it occupies a whole *curve* in spacetime, known as the *worldline* of that object, which is analogous to the notion of a graph of a function in analytic geometry. For instance, in Figure 2, the red line would describe the worldline of a person (let's call her Alice) who is stationary at a position four meters to the right

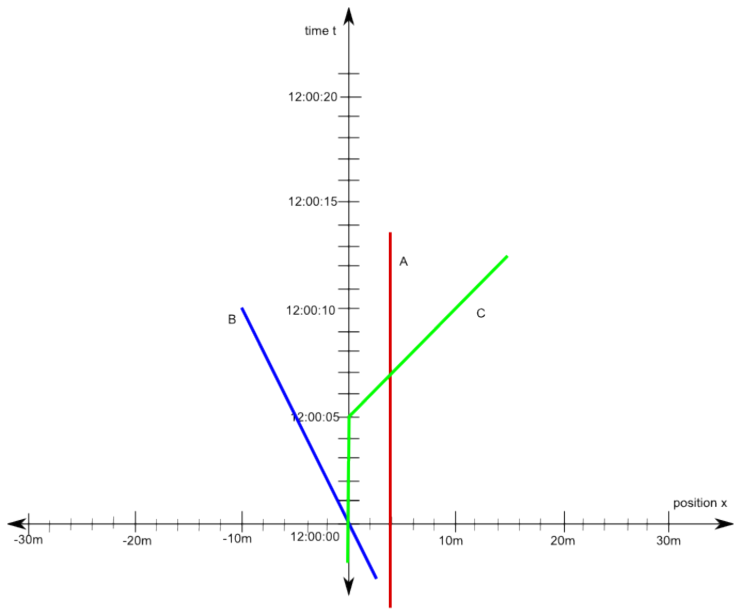

FIGURE 2. The worldlines of Alice, Bob, and Charlie in spacetime.

of the origin); the blue line would describe the worldline of a person (let's call him Bob), who starts at the origin at noon, but is moving at a constant speed of one meter per second to the left; and the green line would describe the worldline of a third person (let's call him Charlie) who is initially stationary at the origin, but five seconds after noon, starts moving at two meters per second to the right.

One advantage of using spacetime diagrams to visualize physical objects is that it becomes easy to see when (or if) two objects pass by each other. For instance, from Figure 2 it is clear that Charlie will pass by Alice at seven seconds after noon.

Exercise 2.1 (Airport puzzle)**.** (Optional) Alice is trying to get from one end of a large airport to another. The airport is 500 meters long, and she walks at one meter per second. In the middle of the airport, there is a moving walkway that is 100 meters long, and moves in the direction Alice wants to go in at the speed of one meter per second; thus, if she is on the walkway while still walking, her net speed will in fact be two meters per second. On the other hand, Alice's shoelaces are undone, and at some point before she arrives at the end of the airport, she has to spend ten seconds to stop walking and tie her shoelaces. She wants to arrive at the other end of the airport as soon as possible, and has to decide whether it is better for Alice to tie one's shoelaces before getting on the walkway, during, or afterwards.

(i) Draw spacetime diagrams to illustrate the three different choices Alice has (tying before, tying during, and tying after the walkway). (An example of the first choice is given in Figure 3.)
(ii) What should Alice do?
(iii) Does the answer to (ii) change if one changes some of the numbers in the exercise (e.g. length of walkway, Alice's walking speed, time taken to tie shoelaces)?

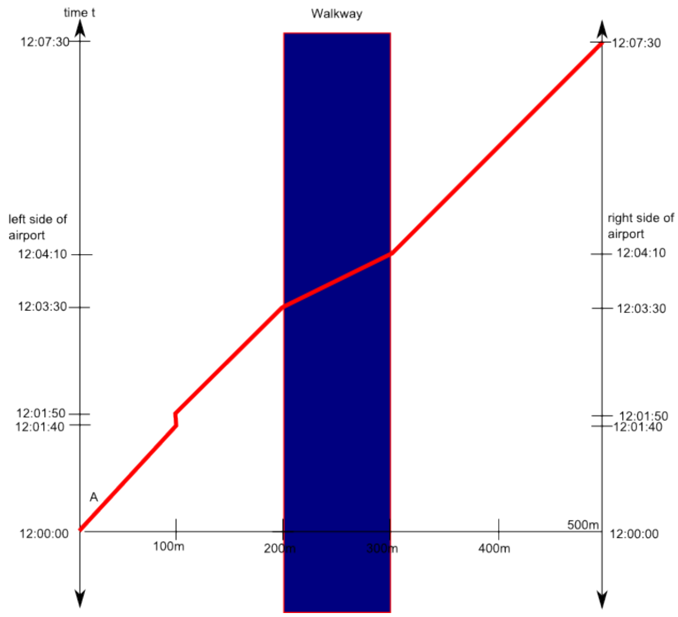

Figure 3. The worldline of Alice at an airport in which she decides to tie her shoelaces before going on the walkway. Note how the walkway and the airport, being objects of significant width, are not represented by a one-dimensional line or curve (or "worldline") but rather by a two-dimensional strip (known as the *worldsheet* of that object).

3. The Doppler effect (in classical physics)

The *Doppler effect*, named after the Austrian physicist Christian Doppler (1803–1853), describes the relation between the observed frequency of a moving object and the velocity of that object. An everyday example of this effect in action comes when hearing the siren of an ambulance or fire engine as it moves past; the frequency (or pitch) of the siren is higher when the ambulance is moving towards the observer, and lower when it is moving away from the observer. This effect can be worked out mathematically, as the following example shows.

Exercise 3.1. An ambulance A, driven by Alice, has a siren that operates at the frequency of 500 Hertz; this means that the sound waves that the siren generates oscillate 500 times a second. We assume that the frequency that the siren emits at remains the same at 500 Hertz, regardless of whether the ambulance is moving or is stationary. It is a calm day with no wind, and sound travels (in any direction) through the air at a speed[1] of 300 meters per second (about 670 miles per hour). An observer Bob (B) is standing still by a road.

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[1]A subtle but important point here: the speed of sound does not depend on the speed of the object emitting that sound; whether the sound originally came from a moving object or a stationary one, it will propagate at 300 meters per second. This is a feature of waves in a medium (in this case, sound waves in air) which stands in contrast to the behavior of particles: an object thrown from a moving platform tends to have a different speed than the same object thrown from a stationary platform. This distinction led to problems with understanding the behavior of light,

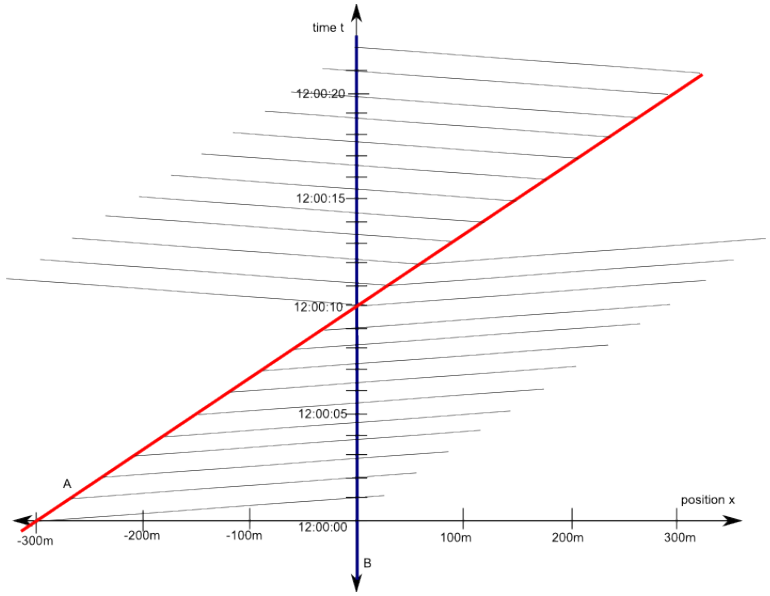

FIGURE 4. The spacetime diagram consisting of a stationary Bob (the blue line) and a moving Alice (the red line), together with sound waves emanating to the left and right from Alice (the black lines). Note how in the bottom half of the diagram (in which Alice is approaching Bob), the sound waves that reach Bob are squashed together (indicating the higher frequency) and in the top half (in which Alice is moving away from Bob) the sound waves that reach Bob are stretched out.

At noon $12:00:00$, Alice is 300 meters away from Bob, but approaching Bob at the speed of 30 meters per second (about 67 miles per hour). Thus, for instance, one second later at $12:00:01$, Alice is now only 270 meters away from Bob, Alice passes Bob at $12:00:10$, and by $12:00:20$, Alice is again 300 meters from Bob but is now receding from Bob at 30 meters per second. (See Figure 4.)

(a) When does the sound emitted at $12:00:00$ reach Bob? When does the sound emitted at $12:00:01$ reach Bob? How many oscillations of the siren will Bob hear in that time interval? What is the frequency that Bob hears of the siren?
(b) When does the sound emitted at $12:00:20$ reach Bob? When does the sound emitted at $12:00:21$ reach Bob? How many oscillations of the siren will Bob hear in that time interval? What is the frequency that Bob hear of the siren?

Now that we have done a specific example, let us now work out the general case. There is nothing special about ambulances and sound waves here; one can apply the Doppler effect analysis to any moving object emitting any sort of radiation.

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which acts both like a wave and a particle, and was part of the puzzle that led to the development of special relativity.

Exercise 3.2 (One-way Doppler shift from Alice to Bob)**.** Suppose that Alice (A) is emitting radiation at some frequency f (i.e. f cycles per second), regardless of whether she is moving or stationary. This radiation travels at speed c in all directions. Another observer Bob (B) is at rest.

(a) If Alice is moving at speed v directly towards Bob (for some $0 < v < c$), at what frequency does Bob perceive Alice's radiation to be? Is this frequency higher or lower than f?
(b) If Alice is moving at speed v directly away from Bob (for some $0 < v < c$), at what frequency does Bob perceive Alice's radiation to be? Is this frequency higher or lower than f?
(c) What happens in (a) or (b) if Alice travels at a speed v that is faster than c? Draw a spacetime diagram to illustrate this situation.

The shift in frequency in (a) and (b) above is known as a (one-way) *Doppler shift*. When the radiation emitted takes the form of light, the Doppler shift is known as a *blue shift* in case (a) (because blue light is a higher frequency than other forms of light) and a *red shift* in case (b) (because red light is a lower frequency than other forms of light).

Now let's do something a bit different: instead of giving Alice the siren, let's give Bob the siren, and see what changes.

Exercise 3.3 (One-way Doppler shift from Bob to Alice)**.** Suppose that Bob (B) is at rest, but is emitting radiation at some frequency f while remaining stationary. This radiation travels at speed c in all directions. Another observer Alice (A) is moving (but agrees with Bob on how fast time is passing). (See Figure 5)

(a) If Alice is moving at speed v directly towards Bob (for some $0 < v < c$), at what frequency does Alice perceive Bob's radiation to be? Is this frequency higher or lower than f? Draw a spacetime diagram to illustrate this situation.
(b) If Alice is moving at speed v directly away from Bob (for some $0 < v < c$), at what frequency does Alice perceive Bob's radiation to be? Is this frequency higher or lower than f? Draw a spacetime diagram to illustrate this situation.

For some reason, the Doppler shift that Bob experiences from Alice's radiation is a bit different from the Doppler shift that Alice experiences from Bob's radiation; there is still blue shift in case (a) and red shift in case (b), but the amount of shift in both cases has changed somewhat. The reason for this is that the situation is not completely symmetrical. From Bob's point of view, the air is stationary, and sound travels at speed c in both directions. But from Alice's point of view, the air is *not* stationary, but is streaming past her at speed v. This changes Alice's perception of the speed of sound; sound goes in front of her at speed only $c - v$ (why?), and behind her at speed $c + v$ (why?). This change in sound speed affects the Doppler shift calculations.

We can combine both Doppler shifts by *reflecting* radiation from Bob to Alice (or vice versa), the way a police radar gun reflects light off of a car.

Exercise 3.4 (Two-way Doppler shifts (optional))**.** Suppose that Bob (B) is emitting radiation at some frequency f while remaining stationary. This radiation travels at speed c in all directions. Another observer Alice (A) is moving. If the

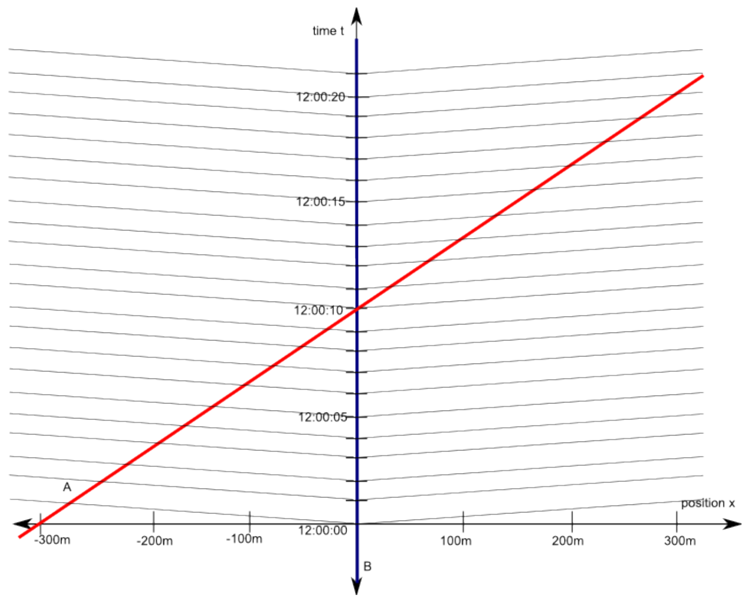

FIGURE 5. The spacetime diagram consisting of a stationary Bob (the blue line) and a moving Alice (the red line), together with sound waves emanating to the left and right from Bob (the black lines). Note how in the bottom half of the diagram (in which Alice is approaching Bob), the sound waves that reach Alice are squashed together (indicating the higher frequency) and in the top half (in which Alice is moving away from Bob) the sound waves that reach Alice are stretched out. However, the stretching and squashing effects here are slightly different from those in Figure 4.

radiation hits Alice, it immediately bounces back in the opposite direction (but still traveling at c), until it returns to Bob.

(a) If Alice is moving at speed v directly towards Bob (for some $0 < v < c$), at what frequency does Bob perceive the reflection of his radiation off of Alice to be? Draw a spacetime diagram to illustrate this situation.
(b) If Alice is moving at speed v directly away from Bob (for some $0 < v < c$), at what frequency does Bob perceive the reflection of his radiation off of Alice to be? Draw a spacetime diagram to illustrate this situation.
(c) What if instead of having the radiation emitting from Bob, bouncing from Alice, and returning to Bob, we consider the reverse situation of radiation emitting from Alice, bouncing from Bob, and returning to Alice?

4. The Doppler effect (in special relativity)

We have seen an asymmetry between the Doppler effect of radiation from Alice to Bob, and from Bob to Alice, in the case when Alice is moving at speed v through the medium of the radiation, and Bob is stationary. For instance, if Alice is moving towards Bob, then Alice's radiation, when received by Bob, is blue-shifted by a factor of $\frac{c}{c-v}$, but Bob's radiation, when received by Alice, is blue-shifted by a factor of $\frac{c+v}{c}$. As we mentioned previously, this asymmetry can be explained by the fact that the medium of radiation (which, in the case of sound waves, is air) is stationary from Bob's point of view, but not from Alice's point of view. On the other hand, the reflected Doppler shift is symmetric: both Alice and Bob perceive a blue shift of $\frac{c+v}{c-v}$ when their emitted radiation is reflected back to them from the other observer.

This asymmetry in one-way Doppler shifts posed a puzzle in the late nineteenth century, when theoretical and experimental physicists attempted to apply this theory to the light emitted by moving bodies. At the time, it was widely believed that light propagated in a hypothetical substance called the *aether*. According to Maxwell's theory of electromagnetism, light should propagate in this aether at a constant speed c in all directions. (This speed c turns out to equal $299,792,458$ meters per second - in fact, nowadays the meter is *defined* so that this is the exact value for c.) On the other hand, no experiment proved capable of distinguishing between whether an observer was moving with respect to the aether or was instead stationary, which suggested among other things that the one-way Doppler shift should actually be symmetric.

The solution to this puzzle was first proposed by Hendrik Lorentz, who postulated that a body A moving at a speed v experiences a *time dilation* effect which slows down all of A's physical reactions by a factor λ, as well as A's perception of time; thus, radiation emitted by A has its frequency slowed down by the factor λ, and any external radiation perceived by A will have its observed frequency sped up by the same factor λ. This makes the one-way Doppler blue shift factor from Bob to Alice $\lambda\frac{c+v}{c}$ instead of $\frac{c+v}{c}$, and the one-way Doppler blue shift factor from Alice to Bob $\frac{1}{\lambda}\frac{c}{c-v}$ instead of $\frac{c}{c-v}$. In order to make the Doppler shifts become symmetric, one must have

$$\lambda\frac{c+v}{c} = \frac{1}{\lambda}\frac{c}{c-v}. \tag{1}$$

This formula - like many other formulae in special relativity - is a little bit messy. One way to simplify it - which is often used by physicists - is to pick a clever choice of units in order to set the speed of light c equal to 1. For instance, instead of working with meters and seconds, one might work with light seconds and seconds (a light second is the amount of distance light travels in a second, i.e. $299,792,458$ meters), so that the speed of light c becomes 1 light second per second. If one chooses these sorts of units, then $c = 1$ and (4) simplifies to

$$\lambda(1+v) = \frac{1}{\lambda}\frac{1}{1-v}. \tag{2}$$

Exercise 4.1. Let Alice move at a speed v while Bob is at rest, where $0 < v < c$.

(a) Let us first suppose one chooses units[2] so that $c = 1$. Solve the equation (2) (making the reasonable assumption that λ is positive) and deduce the formula

$$\lambda = \frac{1}{\sqrt{1-v^2}} \tag{3}$$

[2] The trick of first setting $c = 1$, before considering the general case, is an example of a more general problem-solving strategy in mathematics of considering a simpler special case first, before trying to attack the full problem. This can be a very useful strategy to keep in mind when faced with a complex problem.

for the time dilation experienced by Alice.

(b) Using this value of λ, deduce that the one-way Doppler blue shift from Alice to Bob (or vice versa) is $\sqrt{\frac{1+v}{1-v}}$, when Alice is moving towards Bob, and the units are chosen so that $c = 1$.

(c) Show that the one-way Doppler red shift from Alice to Bob (or vice versa) is $\sqrt{\frac{1-v}{1+v}}$, when Alice is moving away from Bob, and the units are chosen so that $c = 1$.

(d) Now return to the situation in which we do *not* select our units so that $c = 1$. Solve the equation (1) and deduce the formula

$$\lambda = \frac{1}{\sqrt{1 - v^2/c^2}} \tag{4}$$

for the time dilation experienced by Alice.

(e) Deduce that the one-way Doppler blue shift from Alice to Bob (or vice versa) is $\sqrt{\frac{c+v}{c-v}}$, when Alice is moving towards Bob, and we do not choose our units so that $c = 1$.

(f) Show that the one-way Doppler red shift from Alice to Bob (or vice versa) is $\sqrt{\frac{c-v}{c+v}}$, when Alice is moving away from Bob, and we do not choose our units so that $c = 1$.

(g) (optional) How are the relativistic one-way Doppler shifts in (e) and (f) related to the two-way Doppler shifts derived in Exercise 3.4?

The time dilation effect has been observed experimentally in many situations; for instance, radioactive particles take longer to decay when they are accelerated to a speed close to c, by a factor in agreement with the time dilation formula. This effect can also be derived from more fundamental laws of physics (and in particular from a property of these laws known as *Lorentz invariance*), although we will not do so here.

Exercise 4.2. (a) What fraction of the speed of light does Alice has to travel at before her time dilation factor λ becomes 2 (so that Alice does everything twice as slowly as Bob)?

(b) What fraction of the speed of light does Alice has to travel at before her time dilation factor λ becomes 10?

(c) What fraction of the speed of light does Alice has to travel at before her time dilation factor λ becomes 100?

Notice from (4) that if v becomes close to c, then the time dilation factor becomes quite large. For instance, if $v = 0.9c$, then one can compute that $\lambda \approx 2.3$; if $v = 0.99c$ then $\lambda \approx 7.1$; if $v = 0.999c$ then $\lambda \approx 22.4$; and so forth. This leads to an interesting theoretical phenomenon known as the *twin paradox*; if two twins Alice and Bob start off at the same location, but Alice leaves in a spaceship for a while at a large speed (e.g. $v = 0.99c$) for some period of time before returning at the same speed, while Bob remains stationary, then when the twins reunite, Alice will have experienced so much time dilation that she will be significantly younger than Bob. (For instance, it could be that only one year has elapsed for Alice in her journeys, while for Bob over seven years would have passed between Alice's departure and return.) Strictly speaking, the twin paradox is not actually a paradox, because the conclusions drawn above are not logically impossible; but they are highly counterintuitive.

Remark 4.1. The perceptive reader may view the phenomenon of time dilation as introducing the very kind of asymmetry that we were trying to avoid: if Alice is moving and Bob is stationary, couldn't one argue instead that Alice was the one that was stationary, while Bob was the one that was moving, so that Bob should be the one to experience the time dilation, rather than Alice? Actually, it turns out that the situation *is* symmetric (at least until one of Alice or Bob changes velocity): from Bob's point of view, Alice experiences time dilation, while from Alice's point of view, Bob also experiences time dilation. These two claims may initially seem to contradict each other, but they can be reconciled due to another interesting phenomenon in the special theory of relativity called *relativity of simultaneity*. We will not go into the details of this here, but the basic issue is that in order to properly measure the time dilation of an object that is moving through space, one has to *synchronize* one's clocks at different places. This is not an entirely trivial matter, and it turns out that Alice's preferred way of synchronizing clocks and Bob's preferred way of synchronizing clocks do not completely agree with each other, and it is this discrepancy that explains how both Alice and Bob can perceive the other observer to experience time dilation.

5. Mass-energy equivalence

In a famous paper from 1905, Einstein combined his special theory of relativity with other known laws of physics to deduce his famous relationship $E = mc^2$ between the energy content of an object, and the mass of that object. We will now give a (slightly simplified) version of his derivation. It relies[3] on a law from quantum mechanics known as *Planck's relation*[4], which asserts that light consists of particles known as *photons*, with the energy E of each photon (as measured in Joules) being proportional to its frequency f:

$$E = hf,$$

where h is a quantity known as *Planck's constant*[5].

Now imagine that we have a body O of some mass m at rest relative to some observer Bob, which disintegrates[6] into two photons of equal frequency f, one moving to the right, and one moving to the left. We can use Planck's relation, together with the law of conservation of energy, to work out what the original

[3] There are other ways to arrive at $E = mc^2$ that do not use Planck's relation, but this particular derivation is close to Einstein's original arguments, and is quite direct compared to other methods.

[4] Planck's relation can also be written in the form

$$E = \hbar\omega$$

where $\hbar := h/2\pi$ is the *reduced Planck constant* and $\omega := 2\pi f$ is the *angular frequency* (radians per second rather than cycles per second), but we will not use this alternate form of Planck's relation here.

[5] The actual value of h is approximately $6.626 \times 10^{-34} Js$, but this value will not play an important role in the analysis, as we will eventually see all factors of h cancel each other out.

[6] In practice, disintegrations are more complicated than this; for instance, an elementary particle might decay into two other particles plus a photon. But to simplify the discussion we will just consider the disintegration of a mass into two photons; this situation can occur when the mass consists of a particle together with its antiparticle (e.g. an electron-positron pair).

energy content O is, at least from the perspective of Bob. Indeed, Planck's relation tells us that Bob will perceive the two photons to each have an energy of hf, with a total energy of $2hf$; hence, by the law of conservation of energy, the original body O also had an energy content of $2hf$:

$$E = 2hf. \tag{5}$$

Now consider how the situation looks from the perspective of another observer Alice, who is moving at some speed v relative to Bob, either to the right or to the left. Using the Doppler shift formulae from the previous section, we see that the two photons will no longer have a frequency f from Alice's point of view; instead, one of the photons is blue-shifted to frequency $\sqrt{\frac{c+v}{c-v}}f$, and the other is red-shifted to frequency $\sqrt{\frac{c-v}{c+v}}f$. Applying Planck's relation, we see that the total energy of the two photons is

$$h\sqrt{\frac{c+v}{c-v}}f + h\sqrt{\frac{c-v}{c+v}}f$$

and so this is what Alice perceives the energy of the original body O to be.

Now this quantity looks different from Bob's measurement of the energy of the same body O. What explains the discrepancy between these two measurements? Well, from Bob's point of view, the body O is at rest, whereas from Alice's point of view, the object is moving at a velocity v. So it is natural for Alice to assign a higher energy value to O, due to the *kinetic energy* of an object in motion. Using this reasoning, we can compute the kinetic energy K of the body O when moving at speed v by the formula

$$K = h\sqrt{\frac{c+v}{c-v}}f + h\sqrt{\frac{c-v}{c+v}}f - 2hf. \tag{6}$$

This is again a messy formula, but one can simplify the formulae as before by choosing a clever choice of units. We already saw that if we used light seconds instead of meters to measure length (while continuing to measure time in seconds), then we can set the speed of light c equal to 1. In a similar way, if we don't use the Joule as the unit of energy, but rather take the unit energy to be the energy of a photon with a frequency of one Hertz (i.e. one cycle per second), then in these units Planck's constant becomes 1. In these units $c = h = 1$, the equations (5) and (6) simplify to

$$E = 2f \tag{7}$$

and

$$K = \sqrt{\frac{1+v}{1-v}}f + \sqrt{\frac{1-v}{1+v}}f - 2f. \tag{8}$$

Exercise 5.1. (a) Suppose one chooses units so that $h = c = 1$. Use (7) and (8) to deduce the formula

$$K = E\frac{1-\sqrt{1-v^2}}{\sqrt{1-v^2}}.$$

(*Hint:* try to collect all the terms in (8) above a common denominator.) Then rearrange this formula as

$$K = \frac{1}{\sqrt{1-v^2}(1+\sqrt{1-v^2})}Ev^2.$$

(*Hint:* multiply the numerator and denominator by $1 + \sqrt{1 - v^2}$.)

(b) Now suppose that we do not choose units so that $h = c = 1$. Use (5) and (6) to deduce the formula

$$K = E(\frac{1 - \sqrt{1 - v^2/c^2}}{\sqrt{1 - v^2/c^2}}).$$

Then rearrange this formula as

$$K = \frac{1}{\sqrt{1 - v^2/c^2}(1 + \sqrt{1 - v^2/c^2})} \frac{E}{c^2} v^2.$$

This looks like a messy formula for the kinetic energy K, but it (approximately) simplifies[7] in the case when the velocity v is very small compared to c, so that v^2/c^2 is small compared to 1:

Exercise 5.2. Suppose we use the usual units of meters and seconds, so that c is $299,792,458$ meters per second. Suppose that Alice is traveling in the ambulance from Exercise 3.1, in which $v = 30$ meters per second.

(a) Use a calculator to compute the value of $\sqrt{1 - v^2/c^2}$.

(b) Use a calculator to compute the value of $\sqrt{1 - v^2/c^2}(1 + \sqrt{1 - v^2/c^2})$.

More generally, we see that when the velocity v is small (compared to the speed of light), we have

$$\sqrt{1 - v^2/c^2} \approx 1$$

and

$$1 + \sqrt{1 - v^2/c^2} \approx 2$$

and so

$$K \approx \frac{1}{2} \frac{E}{c^2} v^2.$$

On the other hand, from Newtonian mechanics we have the formula

$$K = \frac{1}{2} m v^2$$

and so we expect a similar formula to hold in special relativity when the velocities are small. Comparing the two equations, we arrive at the conclusion

$$\frac{E}{c^2} = m$$

which is Einstein's famous formula $E = mc^2$.

Exercise 5.3. Show that the total energy content of a body of mass m moving at speed v is given by $\frac{mc^2}{\sqrt{1 - v^2/c^2}}$. What happens to this energy as v approaches c? What does this say about the amount of energy required to accelerate a massive body to the speed of light c (or beyond)?

[7] Physicists often use the trick of working with approximations instead of exact formulae in order to simplify the calculations. This can be a very powerful way to solve real-world problems, although if one is not careful and applies an approximate which is not very accurate for the situation at hand, it can give wildly incorrect results. Knowing what approximations are legitimate in any given physical problem requires a certain amount of experience, intuition, and mathematical skill, often requiring advanced mathematical tools such as *differential calculus* or *mathematical analysis*. These topics are unfortunately beyond the scope of the notes, so you will have to take it on faith for now that the approximation used here is justifiable with a bit more mathematical machinery (such as the mathematical concept of a *limit*).

6. Acknowledgments

Thanks to the readers of my blog for supplying corrections, suggestions, and other feedback at `http://terrytao.wordpress.com/2012/12/22` to earlier drafts of this handout.