Finite time blowup for an averaged Navier-Stokes equation

Analysis, PDEs, and Geometry - a conference in honor of Sergiu Klainerman

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The (incompressible) Navier-Stokes equations model the evolution of incompressible fluids such as water. They take the form

$$\partial_t u + (u \cdot \nabla) u = \nu \Delta u - \nabla p$$
$$\nabla \cdot u = 0$$
$$u(0, x) = u_0(x)$$

where $u : [0, +\infty) \times \mathbf{R}^3 \to \mathbf{R}^3$ is the velocity field, $p : [0, +\infty) \times \mathbf{R}^3 \to \mathbf{R}$ is the pressure field, and $u_0 : \mathbf{R}^3 \to \mathbf{R}^3$ is the given initial velocity, and $\nu > 0$ is the viscosity constant. To avoid technicalities we restrict attention to solutions that are smooth and have suitable decay at infinity (we do not work on domains to avoid boundary issues). We also assume the compatibility condition $\nabla \cdot u_0 = 0$.

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Each of the terms in the Navier-Stokes equations

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has a physical interpretation:

- $\nu \Delta u$ represents the dissipative effect of viscosity.
- $(u \cdot \nabla) u$ represents the effect of transport (that the fluid is travelling at velocity u).
- The equation $\nabla \cdot u = 0$ and the compensating pressure term ∇p represent the effects of incompressibility.

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Standard PDE methods (e.g. the energy method) give local existence:

Local existence

If u_0 is smooth and has sufficient decay at infinity, then there exists a time $0 < T_* \leq \infty$ and a smooth solution $u : [0, T_*) \times \mathbf{R}^3 \to \mathbf{R}^3$, $p : [0, T_*) \times \mathbf{R}^3 \to \mathbf{R}$ to the Navier-Stokes equations with initial data u_0. Furthermore, if $T_* < \infty$, the sup norm $\|u(t)\|_{L^\infty(\mathbf{R}^3)}$ goes to infinity as $t \to T_*^-$ (finite time blowup).

One then has the notorious open problem:

Navier-Stokes global regularity problem

Is it always true that $T_* = \infty$?

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- Of course, for physical fluids such as water, the velocity field cannot actually go to infinity, and so the finite time blowup scenario does not occur. So if the answer to the global regularity problem is negative, this means that for certain choices of initial data, the Navier-Stokes equations will at some point cease to be an accurate model for a physical fluid.
- If one works in two spatial dimensions rather than three, the global regularity problem was solved in the 1960s (Ladyshenskaya).
- Why is the three-dimensional problem so much harder than the two-dimensional one?

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- Why is the three-dimensional problem so much harder than the two-dimensional one?

To see the problem heuristically, let us temporarily ignore the role of incompressibility and the pressure p in the Navier-Stokes equations, which then informally become

$$\partial_t u \approx \nu \Delta u - (u \cdot \nabla) u.$$

One can view this equation as a “contest” between the linear heat equation

$$\partial_t u = \nu \Delta u$$

and the transport equation

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then one expects the solution to the Navier-Stokes equation to behave like that of the heat equation

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To get some heuristic understanding of when the viscosity term $\nu\Delta u$ or the transport term $(u \cdot \nabla)u$, suppose that at a given point in time t, the velocity field u achieves an amplitude (i.e. speed) of $A(t)$, and oscillates at a wavelength of $1/N(t)$ (or equivalently, at a frequency of $N(t)$). Then, we expect

$$\nu\Delta u \approx A(t)N(t)^2$$

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$$(u \cdot \nabla)u \approx A(t)N(t)A(t).$$

Thus we expect viscosity domination when $A(t) \ll N(t)$ (amplitude smaller than frequency), and transport domination when $N(t) \ll A(t)$ (amplitude higher than frequency). This heuristic holds in any spatial dimension.

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$$E(t) := \frac{1}{2} \int_{\mathbf{R}^d} |u(t,x)|^2 \, dx$$

is decreasing over time (due to viscosity effects); indeed, from integration by parts we have the energy identity

$$\partial_t E(t) = -\nu \int_{\mathbf{R}^d} |\nabla u(t,x)|^2 \, dx.$$

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If the velocity field u has wavelength $1/N(t)$, then it must be spread out, at minimum, over a ball of radius $\sim 1/N(t)$, which has volume $\sim 1/N(t)^d$. The energy bound

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- In two spatial dimensions $d = 2$, we thus see that the transport-dominated scenario does not occur (the Navier-Stokes equation is critical), and one can show that the viscosity-dominated scenario eventually wins out, which helps explain the result of Ladyshenskaya that one has global regularity of Navier-Stokes in two dimensions.
- But in three dimensions, there is a lot of room between $N(t)^{d/2} = N(t)^{3/2}$ and $N(t)$ (the Navier-Stokes equation is supercritical), allowing for the transport-dominated scenario to occur in this case.
- In general, the regularity problem for supercritical equations are poorly understood; the Navier-Stokes problem is a key test case.

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- The above dimensional analysis suggests an approximately self-similar blowup scenario for three-dimensional Navier-Stokes, in which the amplitude $A(t)$ of the velocity field scales like $N(t)^{3/2}$, and the kinetic energy is concentrated in a ball of radius $\sim 1/N(t)$.
- In this scenario, the transport term dominates, and so the energy should move around at speed $\sim A(t) \sim N(t)^{3/2}$. In particular, it is in principle possible for the energy to concentrate further into a ball of radius $1/2N(t)$ in a time period $\sim 1/N(t)^{5/2}$, increasing the amplitude further to $(2N(t))^{3/2}$ in the process.

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Then, the energy could concentrate further to a ball of radius $1/4N(t)$ in time period $\sim 1/(2N(t))^{5/2}$. Iterating this, one could potentially obtain blowup in time

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This blowup scenario is compatible with the energy identity. But can it actually happen?

I was able to show that blowup is possible if one modifies the equations a little:

Theorem (T., 2014)

There exists an averaged version of the Navier-Stokes equations which obeys the energy identity, but which has solutions that blow up in finite time.

Previous work (Montgomery-Smith, Gallagher-Paicu, Sinai) obtained finite time blowup for variants of Navier-Stokes which did not obey an energy identity.

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Now to explain what averaging means...

We first eliminate the role of the pressure p in the Navier-Stokes equations

$$\partial_t u + (u \cdot \nabla) u = \nu \Delta u - \nabla p$$
$$\nabla \cdot u = 0.$$

Let

$$Pv := v - \Delta^{-1} \nabla (\nabla \cdot v)$$

be the Leray projection (the orthogonal projection to divergence-free vector fields, that eliminates gradients). Applying P, one obtains the Leray form

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The averaged Navier-Stokes equations take the form

$$\partial_t u = \nu \Delta u + \tilde{B}(u, u)$$

where $\tilde{B}(u, u)$ obeys the energy identity

$$\langle \tilde{B}(u, u), u \rangle_{L^2} = 0$$

and is an average of operators of the form

$$T_1 B(T_2 u, T_3 u)$$

where T_1, T_2, T_3 are compositions of (a) rotation operators; (b) dilation operators by scales between 1 and 2; and (c) Fourier multipliers of order 0. These operators are bounded on most standard function spaces (e.g. L^p spaces), so $\tilde{B}$ obeys most of the estimates that B does.

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- Roughly speaking, the theorem shows that one cannot hope to prove global regularity for Navier-Stokes just using the energy identity and estimates for the linear and nonlinear parts of the Navier-Stokes equation.
- This rules out some of the known approaches for establishing regularity. (Big caveat: methods exploiting the vorticity equation are not yet ruled out.)
- Actually, I now believe that finite time blowup for Navier-Stokes is possible, if one carefully chooses the initial data. (More on this later.)

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Now we sketch some details of the proof. The first step is to perform a wavelet decomposition

$$u(t) = \sum_{j,k} X_{j,k}(t)\psi_{j,k}$$

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$$\partial_t u = \nu \Delta u + B(u,u)$$

then becomes an infinite-dimensional system of ODE, roughly of the form

$$\partial_t X_{j,k} = -\nu 2^{2k} X_{j,k} + \sum_{j_1,k_1,j_2,k_2} c_{j,k,j_1,k_1,j_2,k_2} X_{j_1,k_1} X_{j_2,k_2}$$

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- The energy identity then becomes a cancellation condition, namely that the structure constants c_{j,k,j_1,k_1,j_2,k_2} symmetrise to zero after summing over the 3! permutations of (j,k), (j_1,k_1), (j_2,k_2).
- Roughly speaking, an averaged Navier-Stokes equation gives rise to a very similar ODE, but with the structure constants c_{j,k,j_1,k_1,j_2,k_2} replaced with some smaller constants $\tilde{c}_{j,k,j_1,k_1,j_2,k_2}$ that one is at liberty to choose (provided one obeys the above cancellation condition).
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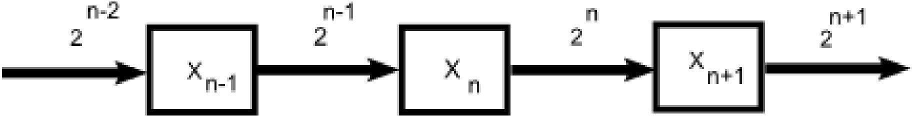
One such model system of ODE was introduced to study the Navier-Stokes equation by Katz and Pavlovic in 2002:

$$\partial_t X_n = -2^{4n/5} X_n + 2^{n-1} X_{n-1}^2 - 2^n X_n X_{n+1}.$$

This can be thought of as a "shell model" of Navier-Stokes, with X_n modeling the energy-normalised velocity $2^{-3n/5} u$ of the fluid at frequencies $\sim 2^{2n/5}$ (in a ball $B(0, O(2^{-2n/5}))$.

Ignoring the viscosity term $-2^{4n/5} X_n$, one can depict this system schematically as follows:



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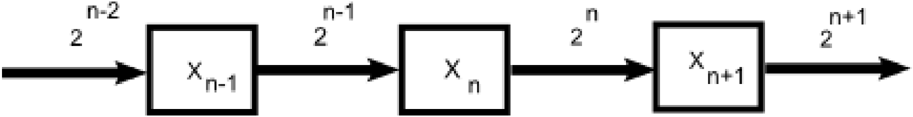
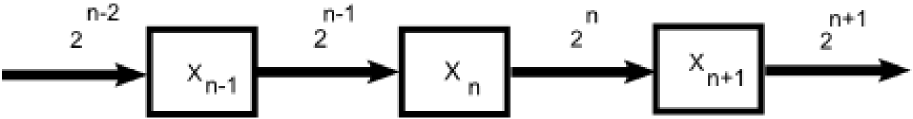
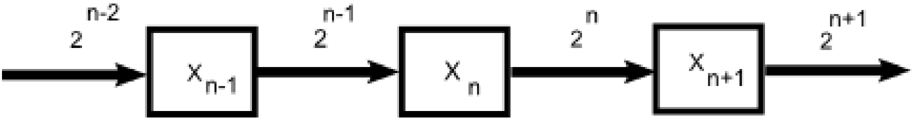
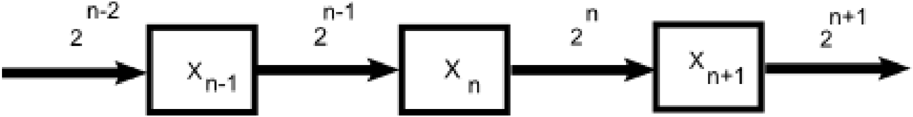




In principle, what should happen with this system is the energy will flow from the X_n mode to the X_{n+1} mode, then to the X_{n+2} mode, at ever faster speeds (corresponding to the blowup scenario in which the energy cascades to higher and higher frequencies at faster and faster rates), until a blowup occurs, with the blowup so fast that the effect of viscosity is negligible.



But it was shown by Barbato, Morandin, and Romito that no blowup occurs. The problem is that as the energy is being transferred from the X_{n-1} mode to the X_n mode, energy is simultaneously being transferred from the X_n to the X_{n+1} mode, and so forth. This leads to a diffusion of the energy into many different modes rather than in just one or two modes, which turns out to decrease the amplitude to the point where the viscosity begins to dominate and prevent the finite time blowup. (But in five and higher dimensions, it was shown by Cheskidov that the viscosity does not dominate, and finite time blowup occurs for the corresponding Katz-Pavlovic models.)



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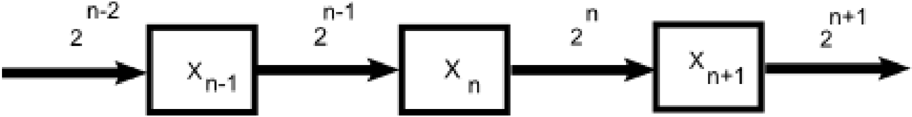

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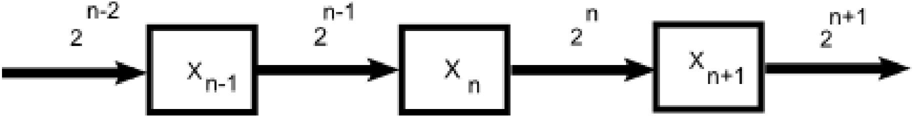

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In order to prevent this diffusion, one needs a system in which a delay is "programmed" between the transfer of energy from the $n-1$ modes to the n modes, and the transfer of energy to the n modes to the $n+1$ modes. It turns out that this can be done using the system

$$\partial_t X_{1,n} = (1+\epsilon_0)^{5n/2}(-\varepsilon^{-2} X_{3,n} X_{4,n} - \varepsilon X_{1,n} X_{2,n} - \varepsilon^2 \exp(-K^{10}) X_{1,n} X_{3,n} + K X_{4,n-1}^2))$$

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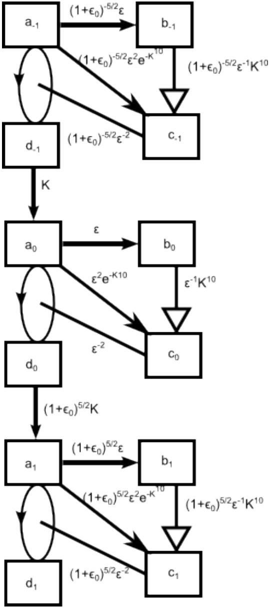
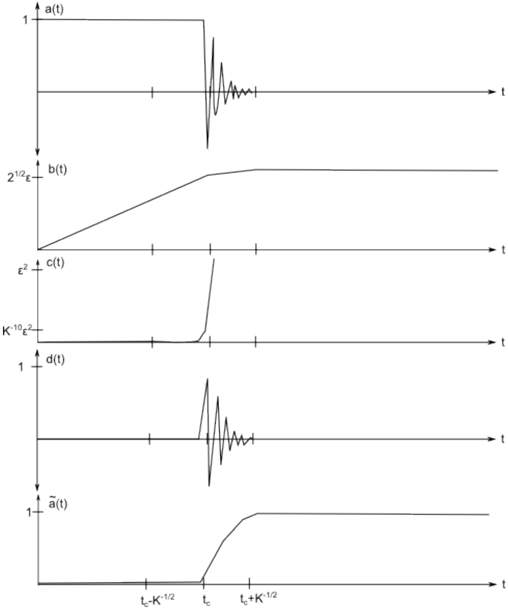

a-1
b-1
c-1
d-1
(1+ϵ0)-5/2ε
(1+ϵ0)-5/2ε2e-K10
(1+ϵ0)-5/2ε-1K10
(1+ϵ0)-5/2ε-2
K
a0
b0
c0
d0
ε
ε2e-K10
ε-1K10
ε-2
(1+ϵ0)5/2K
a1
b1
c1
d1
(1+ϵ0)5/2ε
(1+ϵ0)5/2ε2e-K10
(1+ϵ0)5/2ε-1K10
(1+ϵ0)5/2ε-2


a(t)
1
t
b(t)
$2^{1/2}\varepsilon$
t
c(t)
ε^2
$K^{-10}\varepsilon^2$
t
d(t)
1
t
$\tilde{a}(t)$
1
t
$t_c-K^{-1/2}$
t_c
$t_c+K^{-1/2}$

- Returning back to the fluid interpretation of this system of ODE, the solution evolves as follows.
- At a given point in time, the energy of the solution may be concentrated at a certain frequency scale $(1+\epsilon_0)^n$.
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- The most significant obstacle seems to be the need to build some analogue of logic gates purely out of ideal fluid (as opposed to out of averaged Navier-Stokes equations). With such gates, one can in principle build a Turing-universal computer, and from that one should be able to build the right sort of self-replicating machine.

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Thanks for listening!
Happy birthday, Sergiu!