Correlations of multiplicative functions

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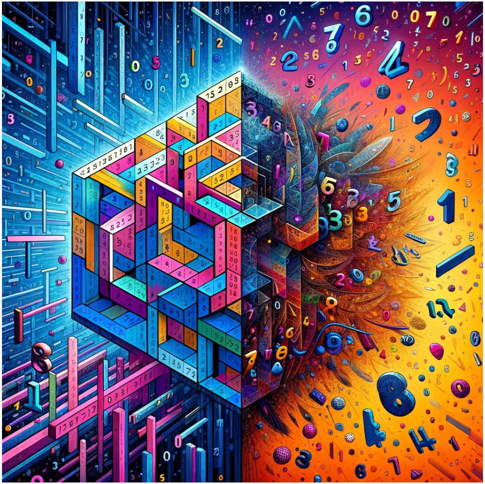
Analytic number theory is often concerned with the distribution of the primes and related objects. I like to view this subject as one of the battlegrounds between **structure** and **randomness**.



Example: twin prime conjecture

Twin prime conjecture

There are infinitely many pairs of primes $p, p+2$.

- Still open, despite centuries of effort.
- From the work of Zhang (2013), Maynard (2013), and Polymath (2014) we know there are infinitely many pairs of primes p, p' with $2 \leq p' - p \leq 246$.
- There is a more general prime tuples conjecture of Hardy and Littlewood that quantifies the number of prime tuples $p + h_1, \ldots, p + h_k$ one should see in a given range.

- Prime number theorem: For large x, there should be roughly $\int_2^x \frac{dt}{\log t}$ primes up to x.
- **Randomness**: If these primes were distributed randomly, one would expect about $\int_2^x \frac{dt}{\log^2 t}$ twin primes up to x, which would solve the twin prime conjecture. (Cramér random model)
- **Structure**: The primes do not distribute completely randomly: for instance, they almost entirely avoid the even numbers, the multiples of 3, and so forth. This leads to the Cramér–Granville random model that makes an improved prediction

$$\left(2\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right)\right)\int_2^x \frac{dt}{\log^2 t} \approx 1.3203\cdots \int_2^x \frac{dt}{\log^2 t}$$

for the number of twin primes up to x.

Numerically, the Cramér–Granville prediction is very accurate!

x	Twin primes up to x	Predicted twin primes up to x
10^4	205	214
10^5	1224	1249
10^6	8169	8248
10^7	58980	58754
10^8	440312	440368
10^9	3424506	3425308
10^{10}	27412679	27411417
10^{11}	224376048	224368865
10^{12}	1870585220	1870559867
10^{13}	15834664872	15834598305
10^{14}	135780321665	135780264894
10^{15}	1177209242304	1177208491861

Unfortunately, we cannot rule out the possibility of *other*, more exotic, structure also being present, which could cause the count of twin primes to deviate from the prediction.

- A major obstruction to establishing results such as the twin prime conjecture is the parity problem.
- To describe this problem, we introduce the Liouville function $\lambda(n)$, defined to equal $(-1)^{\Omega(n)}$ where $\Omega(n)$ is the number of prime factors of n counted with multiplicity.
- Thus, the Liouville function counts the parity of the number of prime factors of n: $\lambda(2) = -1$, $\lambda(6) = +1$, $\lambda(12) = -1$, etc..
- The Liouville function λ is a close cousin of the Möbius function $\mu(n)$ (defined similarly, except that $\mu(n)$ is set to zero if n is not square-free).
- It is a model example of a bounded completely multiplicative function: $\lambda(nm) = \lambda(n)\lambda(m)$ for all n, m.

$$1, -1, -1, 1, -1, 1, -1, -1, 1, 1, -1, -1, -1, 1, 1, 1, -1, -1, \ldots$$

- The Liouville pseudorandomness principle is the heuristic principle that λ behaves (in many statistical senses) like a random sequence of 0s and 1s.
- For instance, the prime number theorem turns out to be equivalent to the asymptotic $\sum_{n \leq x} \lambda(n) = o(x)$ as $x \to \infty$.
- The prime number theorem in arithmetic progressions is equivalent to the asymptotic $\sum_{n \leq x : n = a\ (q)} \lambda(n) = o(x)$ as $x \to \infty$.
- The Riemann hypothesis (RH) is equivalent to the asymptotic $\sum_{n \leq x} \lambda(n) = O(x^{1/2+o(1)})$ as $x \to \infty$.
- The Generalized Riemann hypothesis (GRH) is equivalent to the asymptotic $\sum_{n \leq x : n = a\ (q)} \lambda(n) = O(x^{1/2+o(1)})$ as $x \to \infty$.

- Let us temporarily call a natural number n red if $\lambda(n) = +1$ and green if $\lambda(n) = -1$. The Liouville pseudorandomness principle suggests that red and green numbers are virtually *indistinguishable* in a statistical sense.
- For instance, GRH asserts (roughly speaking) that every long arithmetic progression should contain about the same number of red and green numbers.
- Because of this, most of our analytic number theory techniques (e.g., sieve theory, or the circle method) would not significantly notice if we removed all green numbers (and gave red numbers twice the weight), or vice versa.

- In particular, if one of our techniques could show there were infinitely many twin primes $p, p+2$, they should also be able to show infinitely many twin primes $p, p+2$ where (say) $p+2$ is red (with twice the weight), and not allowed to be green.
- But prime numbers must be green: $\lambda(p+2) = -1$ whenever $p+2$ is prime. So this blocks most analytic number theory methods from proving the twin prime conjecture (the parity barrier).
- To resolve this, one needs techniques that can either "see" the parity (e.g., multiplicative number theory), or transform the problem to one that is not subject to the parity barrier.

The twin prime conjecture is out of reach of current methods, but it has a simpler cousin that looks more tractable:

Chowla's conjecture (1965)

For any distinct integers $h_1, \dots, h_k$, one has

$$\sum_{n \leq x} \lambda(n+h_1) \dots \lambda(n+h_k) = o(x)$$

as $x \to \infty$.

For instance, the $k = 2$ case of Chowla's conjecture implies that

$$\sum_{n \leq x} \lambda(n)\lambda(n+2) = o(x),$$

that is to say that the color of n and $n+2$ are asymptotically uncorrelated. Note for twin primes that $n, n+2$ would both be green.

- Whereas the primes contain both structure and randomness, the Liouville function λ should exhibit “pure randomness”, at least when viewed additively.
- Multiplicative number theory methods - our most powerful method for breaking the parity barrier - do not work directly, because the function $n \mapsto \lambda(n)\lambda(n+2)$ has no obvious multiplicative structure.

- There is a folklore implication: if (a slight generalization) of Chowla's conjecture was true with sufficiently good decay rates in the $o(x)$ term, then one should be able to conclude the twin prime conjecture.
- Unfortunately, the only case of Chowla's conjecture that is currently known is the $k = 1$ case, which is equivalent to the prime number theorem.
- The $k \geq 2$ cases of Chowla's conjecture are still subject to the parity barrier; in some sense, this is the simplest open problem in analytic number theory that has this barrier.

Progress has begun to be made on averaged versions of the Chowla conjecture, in particular:

Logarithmically averaged Chowla's conjecture

For any distinct integers $h_1, \dots, h_k$, one has

$$\sum_{n \leq x} \frac{\lambda(n+h_1)\dots\lambda(n+h_k)}{n} = o(\log x)$$

as $x \to \infty$.

A simple application of summation by parts shows that Chowla's conjecture implies its logarithmically averaged counterpart, but the converse is unclear. However, the logarithmically averaged version is still strong enough for many applications.

- The $k=1$ case $\sum_{n\le x}\frac{\lambda(n)}{n}=o(\log x)$ of the logarithmically averaged conjecture has a short elementary proof (by inspecting partial sums of the identity $\sum_{d|n}\lambda(d)=1_{d\in\mathbf{N}^2}$).
- No similarly short proof of the non-averaged version $\sum_{n\le x}\lambda(n)=o(x)$ (equivalent to the prime number theorem) is known.
- The averaged conjecture is also known for $k=2$ (T. 2015) and odd k (T.–Teraväinen 2017).
- In the $k=2$ case, significant quantitative progress was made by Helfgott–Radziwiłł (2021) and Pilatte (2023).
- The methods can also handle other multiplicative functions than the Liouville function.

One striking application of this machinery is the resolution to an old conjecture of Erdős from 1957:

Erdős discrepancy problem (T., 2015)

For *any* sequence $f : \mathbf{N} \to \{-1, +1\}$, the homogeneous discrepancy

$$\sup_{a,d \in \mathbf{N}} |f(d) + f(2d) + \cdots + f(ad)|$$

is infinite.

Prior to this result, the best lower bound on the supremum was 3 (Konev–Lisitsa 2014), using a massive SAT solver calculation (the initial proof as 13 gigabytes long).

In very brief outline, the relation between the Chowla conjecture and the Erdős discrepancy problem is as follows.

- By multiplicative Fourier analysis, one can (essentially) reduce the discrepancy problem to the case when the function f is completely multiplicative. The task is then to show that the partial sums $f(1) + \cdots + f(d)$ are unbounded. (Polymath, 2012)
- By the van der Corput inequality, it suffices to show that $f(n), f(n + h)$ are asymptotically uncorrelated for any fixed $h > 0$.
- This can be treated by a (generalization of) the $k = 2$ case of the (logarithmically averaged) Chowla conjecture (except when the function f is "pretentious", but this case can be treated separately).

There are fruitful re-interpretations of the Chowla conjecture (and its relatives) using the language of dynamical systems and ergodic theory. For instane, it is known (Sarnak 2015, T., 2016) that the logarithmic Chowla conjecture is equivalent to a conjecture relating the Möbius function to dynamics:

Logarithmic Sarnak conjecture

Let $T : X \to X$ be any zero entropy homeomorphism on a compact space X. Then for any continuous $F : X \to \mathbf{C}$ and any $x_0 \in X$, one has $\sum_{n \leq x} \frac{\lambda(n) F(T^n x_0)}{n} = o(\log x)$ as $x \to \infty$.

For instance, $\sum_{n \leq x} \frac{\lambda(n) e^{2\pi i P(n)}}{n} = o(\log x)$ for any polynomial P. Many, many cases of the Sarnak conjecture (with or without logarithmic averaging) are now known; but the general case remains out of reach.

There is an emerging, and partially successful, strategy to attack the (logarithmic) Chowla conjecture, as follows.

- By using the multiplicativity $\lambda(pn) = -p\lambda(n)$ of the Liouville function for small primes p, combined with tools from probability and additive combinatorics, relate the Chowla conjecture to behavior of *short averages* such as $\sum_{x \leq n \leq x+H} \lambda(n)$ or $\sup_\alpha |\sup_{x \leq n \leq x+H} \lambda(n) e^{2\pi i \alpha n}|$, with H very small compared to x.

- By using multiplicativity at larger primes, and further techniques from analysis and combinatorics, relate these short averages to long averages (where H is close to x).
- Use techniques from multiplicative number theory, such as contour integration, to control the long averages.

From correlations to short averages

- One can use multiplicativity at small primes to write sums such as

$$\sum_{n \leq x} \frac{\lambda(n)\lambda(n+1)}{n}$$

as sums of the form

$$\sum_{n \leq x} \frac{\lambda(n)\lambda(n+p)}{n} p 1_{p|n}$$

up to small errors.

- If the correlations $\lambda(n)\lambda(n+p)$ do not themselves correlate with divisibility of n by p, one can hope to approximate the sum

$$\sum_{n \leq x} \frac{\lambda(n)\lambda(n+p)}{n} p 1_{p|n}$$

by

$$\sum_{n \leq x} \frac{\lambda(n)\lambda(n+p)}{n}.$$

Averaging over primes $p \sim H$, one arrives at short averages that morally look like

$$\int_2^x |\frac{1}{H} \sum_{t \leq n < t+H} \lambda(n)|^2 \frac{dt}{t}.$$

- This strategy can be made rigorous through entropy methods (T. 2015, Teraväinen–T. 2018), and more recently (for $k = 2$) by expander graph methods (Helfgott–Radziwiłł 2021, Pilatte 2023).
- For higher k, more complicated averages such as

$$\int_2^x (\sup_\alpha |\frac{1}{H} \sum_{t \leq n < t+H} \lambda(n) e^{2\pi i \alpha n}|)^2 \frac{dt}{t}$$

appear (and for $k \geq 4$, higher order Fourier analysis is needed).

From short averages to long averages

- To move from short averages to longer averages, one can use the contour integration techniques of Matomäki–Radziwiłł (2015) for the $k = 2$ case.
- For higher k, one can use a classical distribution inequality of Elliott, together with combinatorial graph theory methods of Matomäki–Radziwiłł–T. (2018), Matomäki–Radziwiłł–T.–Teräväinen-Ziegler (2002), the contagion argument of Walsh (2023), or the pyramid argument of Walsh (2023).

A very quick sketch of the contagion argument:

- If λ correlates with $n \mapsto e^{2\pi i \alpha(x) n}$ on an interval $[x, x+H]$, then by the Elliott inequality, it also correlates with $n \mapsto e^{2\pi i \alpha(x) n}$ on the multiples of p in $[x, x+H]$ for many primes $p \sim P$.
- By multiplicativity, this implies that λ correlates with $n \mapsto e^{2\pi i p \alpha(x) n}$ on $[x/p, x/p + H/p]$ for many $p \sim P$.
- By the large sieve inequality, this creates relations $p\alpha(x) \approx p'\alpha(x')$ for many $x, x' \sim X$ and $p, p' \sim P$ with $x/p \approx x'/p'$.

- By some combinatorics and elementary number theory, this implies that there are many larger intervals $[x, x + PH]$ with $x \sim PX$ such that $\alpha(x/p) = p\alpha'(x)$ for some frequency $\alpha'(x)$: the frequency structure on length H intervals has "infected" the length PH intervals.
- By iterating this process, one can get from structure on short intervals to structure on large intervals.

For $k = 2$, we have managed to obtain increasingly strong estimates, e.g., one can bound $\sum_{n \leq x} \frac{\lambda(n)\lambda(n+1)}{n}$ by

- $O(\log x)$ (trivial bound)
- $O(\log x / (\log\log\log\log x)^{1/5})$ (T. 2016)
- $O(\log x / (\log\log\log x)^{c})$ (Teräväinen–T. 2018)
- $O(\log x / (\log\log x)^{1/2})$ (Helfgott–Radziwiłł 2021)
- $O(\log x / \log^{c} x)$ (Pilatte 2023)

Removing the final logarithm is on the same order of difficulty as the Riemann hypothesis!

- For odd k there is a parity trick specific to the Liouville function, based on exploiting the negative sign in the identity

$$\lambda(n+h_1)\dots\lambda(n+h_k) = -\lambda(pn+ph_1)\dots\lambda(pn+pn_k),$$

  but it does not generalize well to arbitrary multiplicative functions.
- In other cases we are currently experiencing a *gap* in the strategy: small prime multiplicativity can relate the Chowla conjecture to short averages (at scales less than $\log^\varepsilon x$ for a small ε), but large prime multiplicativity methods can only control slightly longer averages (at scales larger than $\log^A x$ for a large A).
- Recent work of Walsh (2023) hints though that this gap might be bridgeable. Stay tuned...

Thanks for listening!