Translational tilings of Eucliden space

Terence Tao (joint with Rachel Greenfeld)

University of California, Los Angeles

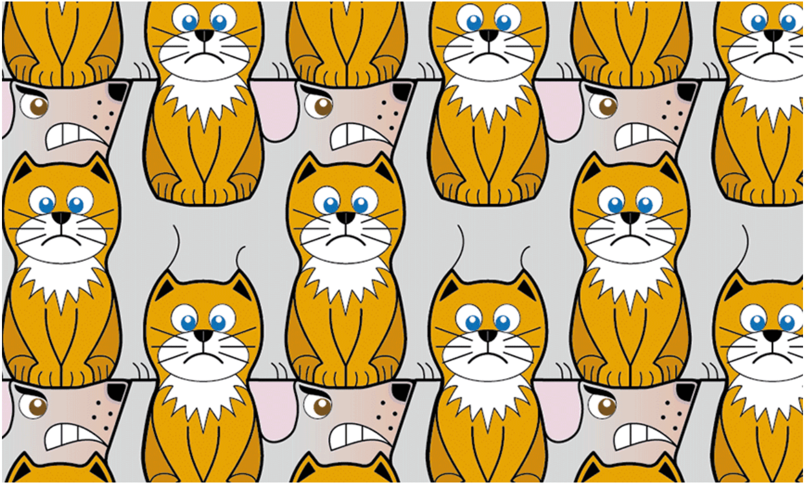
Jan 4, 2024

Tiling a group by translations

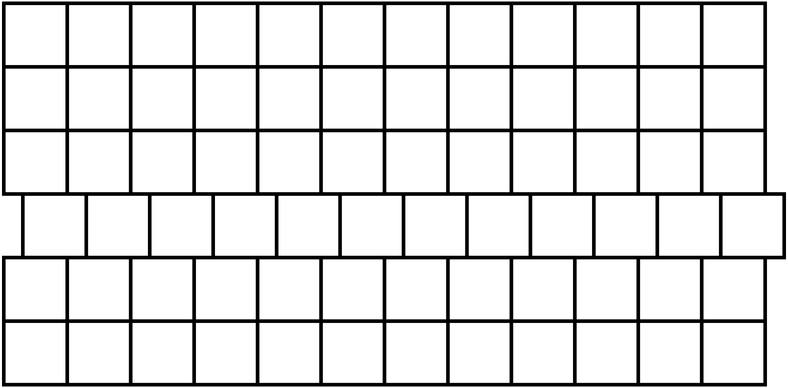
- Let $G = (G, +)$ be an additive group (such as a lattice $\mathbf{Z}^d$ or Euclidean space $\mathbf{R}^d$).
- A translational tiling (or *tiling*, for short) of G by a tile $F \subset G$ is a partition (possibly up to null sets) of G into disjoint translates $a + F, a \in A$ of F along some tiling set A. When this occurs we write $A \oplus F = G$.
- If there is a lattice Λ (a discrete cocompact subgroup of G) such that A is Λ-periodic ($A + \lambda = A$ for all $\lambda \in \Lambda$), we say that the tiling is periodic.

- One can also consider tilings $(A_1 \oplus F_1) \uplus \cdots \uplus (A_J \oplus F_J) = G$ of G by multiple tiles $F_1, \ldots, F_J$, rather than just one tile. (Tilings by one tile are sometimes known as monotilings, to distinguish them from tilings by multiple tiles.)
- In the literature one also frequently considers monohedral tilings (which involve rotations or reflections as well as translations), but in this talk we will focus exclusively on translational tilings.

A periodic monotiling $A \oplus F = \mathbf{R}^2$ of the plane by a pegasus F.
[Escher, 1959]



A periodic tiling $(A_1 \oplus F_1) \uplus (A_2 \oplus F_2) = \mathbf{R}^2$ of the plane by a cat F_1 and a dog F_2. [Nicolas, 1999]



A non-periodic monotiling $A \oplus F = \mathbf{R}^2$ of $\mathbf{R}^2$ by a square F.

Connections to other fields

- Translational monotilings $A \oplus F = G$ have additional algebraic structure that other tilings do not, making them significantly more "rigid".
- One hint of this is seen by writing the tiling equation $A \oplus F = G$ in convolution form

$$1_A * 1_F = 1,$$

which looks tempting to analyze via the Fourier transform.

The tiling equation $A \oplus F = G$ also exhibits some unexpected symmetries.

- Reflection symmetry: if $A \oplus F = G$, then $A \oplus -F = G$. This has a short combinatorial proof (based on the obvious algebraic fact that $a + f = a' + f' \iff a - f' = a' - f$).
- Dilation symmetry: if G is discrete, $A \oplus F = G$, and p is coprime to $|F|$, then $A \oplus pF = G$. This has a short elementary number-theoretic proof (based on the Frobenius identity $(a + b)^p = a^p + b^p$ in any commutative ring of characteristic p).

- Given a tiling F of a discrete group G, the space of tilings $\{A : A \oplus F = G\}$ can be viewed as a compact space that is invariant under the action of the group G by translation; that is to say, it is a topological dynamical system.
- It is then tempting to analyze tilings through the lens of dynamics and ergodic theory.

- Despite all this structure, translational tilings can still exhibit remarkably complex behavior, particularly in higher dimensions.
- There are now several examples of conjectures about translational tilings that are true (or suspected to be true) in low dimensions, but fail in higher dimensions. (Keller's conjecture, Fuglede's conjecture, the periodic tiling conjecture, ...)
- As we shall see, the deceptively simple-looking equation $1_A * 1_F = 1$ can have an incredibly rich solution space!

Vague questions

Let F be a subset of an additive group G.

- Can one determine whether F actually tiles G? That is to say, whether the tiling equation $A \oplus F = G$ has a solution A?
- What can one say about the structure of the tiling sets A associated to a tile F?

More precise questions

Let F be a finite subset of a discrete group G such as $\mathbf{Z}^d$, or a bounded measurable subset of $G = \mathbf{R}^d$.

- **Logical decidability:** Is the assertion that F tiles G (i.e., that there is a solution to $A \oplus F = G$) a decidable or undecidable sentence in ZFC?
- **Algorithmic decidability:** Is there an algorithm that, given F and E as input, will determine in finite time whether F tiles G?
- **Periodic tiling conjecture:** [Stein 1974, Grunbaum–Shephard, 1987; Lagarias–Wang, 1996] If F tiles G, does this mean that F also tiles G *periodically*? In other words, can every non-periodic tiling be "repaired" to be periodic?

Logical decidability implies **algorithmic decidability**! (In discrete cases, at least.)

- Suppose that for each F, the question of whether F tiles G is **logically decidable**.
- Then we claim that this question is also **algorithmically decidable** for arbitrary inputs F. (Assuming of course that ZFC is consistent.)
- The algorithm is simple: get a computer to search in parallel for proofs or disproofs of the assertion that F tiles G.
- By logical decidability, this algorithm will halt in finite time and will determine whether F tiles G or not.

(Side remark: the same argument shows that the Gödel incompleteness theorem is a corollary of the undecidability of the halting problem for Turing machines.)

The **periodic tiling conjecture** implies **logical decidability** (and hence **algorithmic decidability**)! [Wang, 1966]

- If F tiles G periodically, then one can use this periodic tiling to give a finite length proof (in ZFC) that F tiles G.
- If F fails to tile G, then (by the compactness theorem) there is some finite window $G \cap W$ of G that cannot be tiled by F (in the sense that there is no A with $(A \oplus F) \cap W = G \cap W$, and this can be converted to a finite length proof (in ZFC) that F does not tile E.
- If the **periodic tiling conjecture** holds, then the above two cases are the only possible cases.

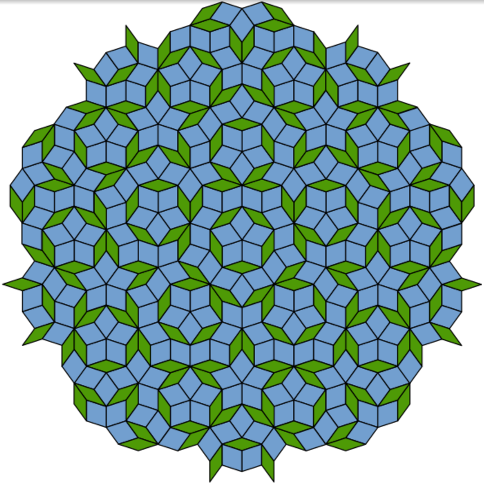
So it is of interest to settle the **periodic tiling conjecture**.

Some positive progress towards the **periodic tiling conjecture**:

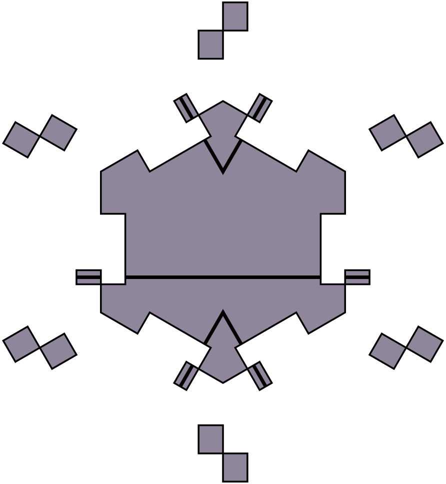
- Trivial when G is a finite abelian group (since all subsets of G are trivially periodic).
- True when $G = \mathbf{Z}$ [Newman, 1977], $G = \mathbf{R}$ [Lagarias–Wang, 1996], or $G = \mathbf{Z} \times G_0$ for a finite G_0 [Greenfeld–T., 2021].
- True when $G = \mathbf{Z}^2$ [Bhattacharya, 2020, Greenfeld–T., 2020].
- True when $G = \mathbf{Z}^d$ and $\#F$ is a prime, or at most 5 [Szegedy, 1998].

- However, once the number of tiles J is large, the **periodic tiling conjecture** breaks down.
- That is to say, there exist aperiodic tilings - a set of J tiles that tile a group non-periodically, but cannot tile that group periodically.

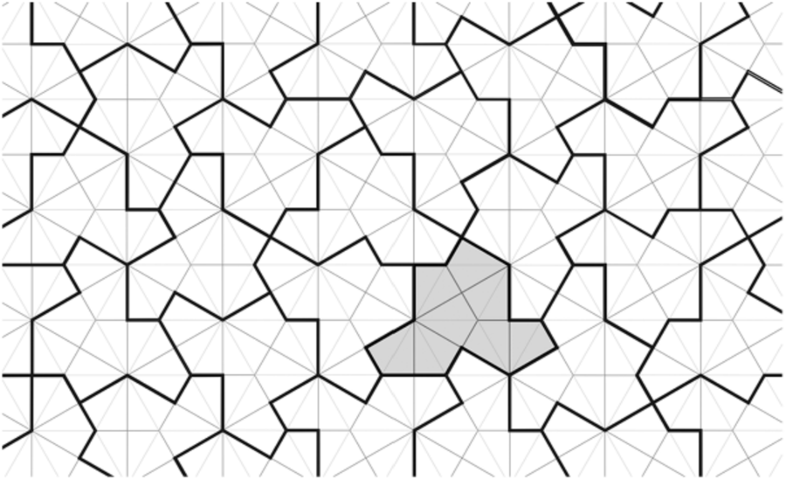
- In fact, the **logical decidability** and **algorithmic decidability** of tilings also fail for large J [Berger, 1966]!
- For instance, there are finite subsets $F_1, \dots, F_J$ of $\mathbf{Z}^2$ that tile $\mathbf{Z}^2$, but for which it is not possible to prove (or disprove) in ZFC that they do so.
- (Logically) undecidable tilings are necessarily aperiodic [Wang, 1966], but the converse is not true in general.



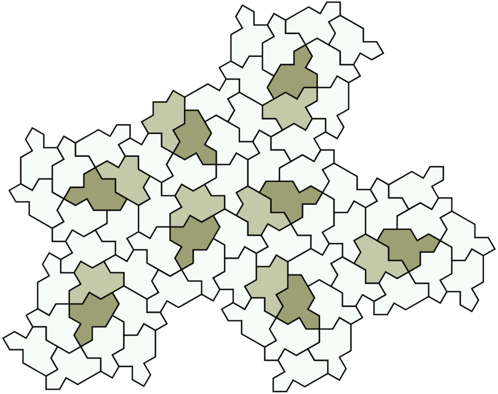
A Penrose tiling, which is perhaps the best known example of an aperiodic tiling of $\mathbf{R}^2$ (in this case, with $J = 10$ tiles, which are rotations of two types of rhombi). [Penrose, 1974; Ammann, 1976]



The Socolar–Taylor tile. The twelve copies of this tile formed by rotating and reflecting by the symmetries of the regular hexagon can tile $\mathbf{R}^2$, but only aperiodically. [Socolar–Taylor 2010]



The hat tile. This is a topological disk (connected with no holes) which has the same aperiodicity property as the Socolar–Taylor tile. [Smith, Myers, Kaplan, and Goodman–Strauss 2023]



The spectre tile. This has similar aperiodicity properties to the hat tile, except that no reflections are needed in order to tile the plane. [Smith, Myers, Kaplan, and Goodman–Strauss 2023]

High dimensional tilings turn out to be rather pathological!

Theorem

- (Greenfeld–T., 2022) For d sufficiently large, the periodic tiling conjecture is false in $\mathbf{R}^d$ and $\mathbf{Z}^d$.
- (Greenfeld–Kolountzakis, 2023) The conjecture remains false in high dimensions even if one requires that the tile is connected.
- (Greenfeld–T., 2022) There exists a finite abelian group G_0 such that the periodic tiling conjecture is false in $\mathbf{Z}^2 \times G_0$.
- (Greenfeld–T., 2023) The problem of determining whether a finite set F tiles (a periodic subset of) $\mathbf{Z}^d$ is both algorithmically undecidable and logically undecidable for d sufficiently large.

- Our threshold for “sufficiently large” is explicitly computable, but enormous.
- It remains open whether the periodic tiling conjecture, logical decidability, or algorithmic decidability could still be true in $\mathbf{Z}^3$ (or in $\mathbf{R}^2$), possibly with additional hypotheses on the tile F.
- For this talk we will focus on disproving the periodic tiling conjecture in $\mathbf{Z}^2 \times G_0$. For the more recent results on undecidability, see Greenfeld’s talk later this afternoon!

High-level summary of proof:

- A counterexample in $\mathbf{Z}^2 \times G_0$ for some finite abelian G_0 can be constructed from a counterexample to a "multiple periodic tiling conjecture" in $\mathbf{Z}^2 \times G_1$ for some finite abelian G_1.
- The multiple tiling problem creates a "tiling language" that can express many other constraints, and in particular can set up a "q-adic Sudoku puzzle".
- We construct a q-adic Sudoku puzzle that is solvable only with non-periodic solutions, to generate the required counterexamples.

From a single tiling equation to a system

- Our goal is to construct a finite set $F \subset \mathbf{Z}^2 \times G_0$ for some G_0 such that solutions to the tiling equation $F \oplus A = \mathbf{Z}^2 \times G_0$ exist, but are all non-periodic.
- It turns out that it suffices to construct a collection $F_1, \ldots, F_M \subset \mathbf{Z}^2 \times G_1$ of finite sets for some G_1 such that solutions to the tiling system

$$F_m \oplus A = \mathbf{Z}^2 \times G_1 \text{ for } m = 1, \ldots, M$$

  exist, but are all non-periodic.
- This is a much more flexible framework that will make it easier to construct counterexamples.

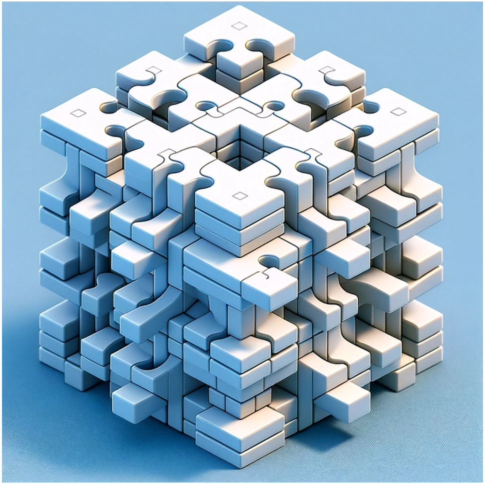
The basic idea is to combine all of the tiles $F_m \subset \mathbf{Z}^2 \times G_1$ in the tiling system

$$F_m \oplus A = \mathbf{Z}^2 \times G_1 \text{ for } m = 1, \dots, M$$

into a single "sandwich"

$$F := \bigcup_{m=1}^{M} F_m \times E_m$$

in a larger group $\mathbf{Z}^2 \times G_0 = \mathbf{Z}^2 \times G_1 \times \mathbf{Z}/N\mathbf{Z}$, for suitably chosen sets $E_m \subset \mathbf{Z}/N\mathbf{Z}$.



If the "sandwich" F is chosen correctly, there will be a correspondence between solutions of the system $F_m \oplus A = \mathbf{Z}^2 \times G_1$, $m = 1, \dots, M$ and solutions of the single equation $F \oplus A = \mathbf{Z}^2 \times G_0$.

Tiling language

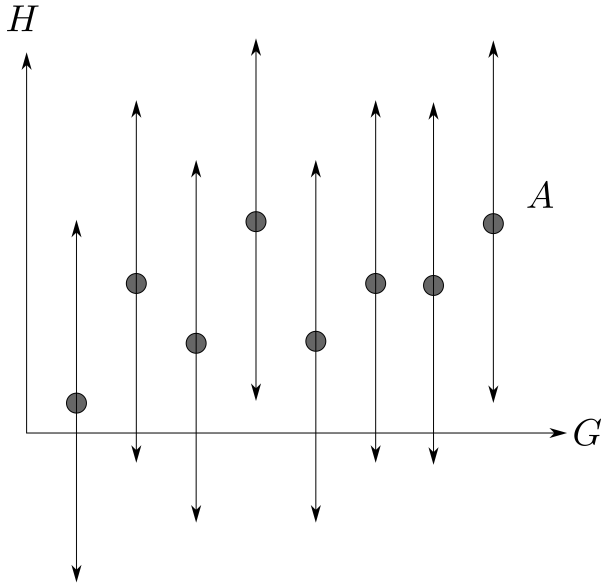
- We view the individual equations $F_m \oplus A = \mathbf{Z}^2 \times G_1$ of a system of tiling equations as a sentence in a tiling language that gives some constraints on the set A.
- Our strategy is to explore this tiling language and see what kinds of constraints we may place on such sets A using this language.
- We aim to build up a “library” of useful constraints of this type, much as how a programming language can develop a library of useful subroutines.
- Then, we use this library to “program” an interesting system of constraints that admits only non-periodic solutions.

Sample sentence 1: “I am a graph”

For instance, the assertion that a set $A \subset G \times H$ is a graph $A = \{(x, f(x)) : x \in G\}$ of some function $f : G \to H$ can be encoded as the tiling equation

$$(\{0\} \times H) \oplus A = G \times H;$$

this is a fancy way of writing the vertical line test from an undergraduate math course.

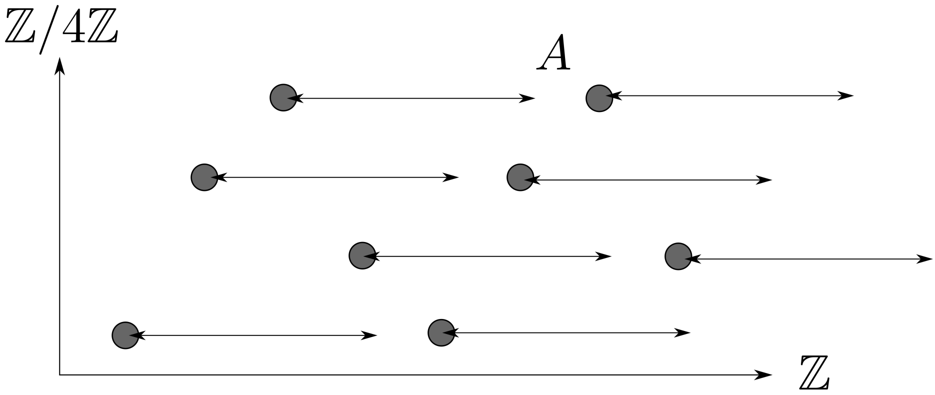

H
A
G

Sample sentence 2: “I am a periodized permutation”

We have seen how to express the property of a set $A \subset \mathbf{Z} \times \mathbf{Z}/N\mathbf{Z}$ being the graph $\{(x, f(x)) : x \in \mathbf{Z}\}$ of a function $f : \mathbf{Z} \to \mathbf{Z}/N\mathbf{Z}$.

We can encode the additional property that f takes the form $f(x) = \sigma(x \bmod N)$ for some permutation $\sigma : \mathbf{Z}/N\mathbf{Z} \to \mathbf{Z}/N\mathbf{Z}$ by the additional tiling equation

$$(\{1, \dots, N\} \times \{0\}) \oplus A = \mathbf{Z} \times \mathbf{Z}/N\mathbf{Z}.$$


$\mathbb{Z}/4\mathbb{Z}$
A
$\mathbb{Z}$

With other constructions like this, one can (modulo some technicalities) encode sentences such as

- "f is a function that is periodic in a given direction v".
- "f is a function that is boolean in that it takes only two values, such as $\{0, 1\}$."
- "$f_1, \dots, f_M$ are boolean functions that jointly take values in some specified subset Ω of $\{0, 1\}^M$."

This is an expressive enough library of sentences to start encoding Sudoku-type puzzles.

Standard Sudoku

9	4	2	1	6	3	8	5	7
5	3	6	2	8	7	9	4	1
8	7	1	9	5	4	2	3	6
3	2	7	8	1	9	4	6	5
1	5	4	3	2	6	7	9	8
6	9	8	7	4	5	1	2	3
2	6	5	4	7	1	3	8	9
7	8	9	6	3	2	5	1	4
4	1	3	5	9	8	6	7	2

Recall that a standard solution to a Sudoku puzzle consists of a function from a 9×9 board to a set of digits $\{1, \dots, 9\}$, such that the restriction of the function to any row, column, or 3×3 block is a permutation.

q-adic Sudoku

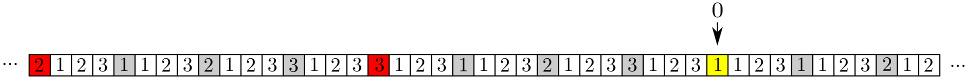
⋮

3	2	1	3	3	2	1	2	3	2	1	1	3	2	1	2
2	1	3	3	2	1	2	3	2	1	1	3	2	1	1	3
1	3	3	2	1	2	3	2	1	1	3	2	1	1	3	2
3	3	2	1	2	3	2	1	1	3	2	1	3	3	2	1
3	2	1	2	3	2	1	1	3	2	1	2	3	2	1	3
2	1	2	3	2	1	1	3	2	1	1	3	2	1	3	3

⋮

We introduce the notion of a "q-adic Sudoku solutions" for a given base q (which for technical reasons we take to be a power of two), which is a function from a board with finitely many columns and infinitely many rows to $\{1, \dots, q-1\}$, which has a certain structure on every row and diagonal (and more generally any line of integer slope).

Using the tiling language described earlier, one can identify q-adic Sudoku solutions with the solution A to a certain system of tiling equations $F_m \oplus A = \mathbf{Z}^2 \times G_1$.



To define the Sudoku puzzle more precisely, we introduce the "q-adic function" $f_q : \mathbf{Z} \to \{1, \dots, q-1\}$, defined by $f_q(n) = a$ when $n = q^i(qm + a)$ for some integers i, m and $a = 1, \dots, q-1$ (and with the convention $f_q(0) = 1$). In other words, $f_q(n)$ is the final non-zero digit in the base q expansion of n. This function is almost periodic (a limit in density of periodic functions), but not actually periodic.

$$\vdots$$

3	2	1	3	3	2	1	2	3	2	1	1	3	2	1	2
2	1	3	3	2	1	2	3	2	1	1	3	2	1	1	3
1	3	3	2	1	2	3	2	1	1	3	2	1	1	3	2
3	3	2	1	2	3	2	1	1	3	2	1	3	3	2	1
3	2	1	2	3	2	1	1	3	2	1	2	3	2	1	3
2	1	2	3	2	1	1	3	2	1	1	3	2	1	3	3

$$\vdots$$

A *Sudoku solution* is a function F on a half-infinite board $\{1, \ldots, q^2\} \times \mathbf{Z}$ to $\{1, \ldots, q-1\}$ whose restriction $n \mapsto F(n, jn+i)$ to any line of integer slope is a rescaled version $n \mapsto c_{i,j} f_q(a_{i,j} n + b_{i,j})$ mod q of f_q for some integer coefficients $c_{i,j}, a_{i,j}, b_{i,j}$.
The Sudoku solution is said to have *good columns* if every column agrees with a q-periodized permutation, outside of a single coset of $q\mathbf{Z}$.

The key proposition in our argument is then

Proposition (Greenfeld–T., 2022)

Let q be a sufficiently large power of 2. Then there exist q-adic Sudoku solutions with good columns, but they are all non-periodic.

Because q-adic Sudoku solutions can be identified with solutions to certain system of tiling equations in $\mathbf{Z}^2 \times G_1$, this allows us to contradict the periodic tiling conjecture.

First step: approximate affineness

There are two main steps in the proposition.
By definition, q-adic Sudoku solutions are "approximately affine" along all rows and diagonals. It turns out that this implies (for q large enough) that such solutions are also "approximately affine" in a two-dimensional sense. This is a variant of the following simple result, which we leave to the audience:

Proposition

Suppose that $F : \mathbf{R}^2 \to \mathbf{R}$ is a function which is affine on every horizontal line $\{(x, y) : y = c\}$, diagonal $\{(x, y) : y - x = c\}$ and antidiagonal $\{(x, y) : y + x = c\}$ for all $c \in \mathbf{R}$. Then F is affine (i.e., $F(x, y) = Ax + By + C$ for some reals A, B, C).

Note that we need all three families of lines to give the proposition, otherwise there are counterexamples such as $F(x, y) = y(y - x)$.

Normal form

$\vdots$

$\cdots$

2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	2	3	1	1	2	3	2	1	2	3	3	1	2	3	1	1	2	3	2	1	2	3
3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

$\cdots$

$\vdots$

Once one establishes that q-adic solutions are approximately affine, there is a normalization available to place them in a "normal form" in which they are essentially constant on all rows outside of those indexed by a coset of $q\mathbf{Z}$.

Second step: “Tetris”

⋮

	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	
…	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	…
	1	2	3	1	1	2	3	2	1	2	3	3	1	2	3	1	1	2	3	2	1	2	3	

⋮

If one deletes all rows outside of the exceptional coset (similar to how the game “Tetris” deletes rows that have been completely filled), one ends up with a new q-adic Sudoku solution that is related to the previous one.

By carefully analyzing this new solution and iterating the process, one can establish non-periodicity of such solutions.

Thanks for listening!