Proof of the Sequence Representation in the Bounded Power Series

Problem Statement

Let $a_0, a_1, \ldots$ be a bounded sequence of real numbers, and suppose that the power series

$$f(x) := \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$$

decays like $O(e^{-x})$ as $x \to \infty$, in the sense that $e^x f(x)$ remains bounded as $x \to \infty$. The problem is to show that $a_n = C(-1)^n$ for some constant C.

Proof

Step 1: Converse Statement

If $a_n = C(-1)^n$ for some C, then the power series $f(x)$ can be written as:

$$f(x) = \sum_{n=0}^{\infty} C(-1)^n \frac{x^n}{n!} = C \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!} = Ce^{-x}.$$

This clearly shows that $f(x)$ decays like $O(e^{-x})$ as $x \to \infty$.

Step 2: Introducing the Laplace Transform

Suppose that a_n is a bounded sequence of real numbers with $f(x) = O(e^{-x})$ as $x \to \infty$. We introduce the Laplace transform

$$\mathcal{L}f(s) := \int_0^{\infty} f(x) e^{-sx} \, dx.$$

The decay hypothesis $f(x) = O(e^{-x})$ ensures that the Laplace transform is well-defined for $\Re(s) > -1$ and is holomorphic in this half-plane.

Step 3: Bound on the Laplace Transform

Using the decay hypothesis, we can show that

$$\mathcal{L}f(s) = O\left(\frac{1}{\Re(s)+1}\right)$$

for $\Re(s) > -1$.

Step 4: Power Series Representation

Using the power series expansion

$$f(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!},$$

we find that in the region $\{s : \Re(s) > -1, |s| > 1\}$, the Laplace transform has the alternate form

$$\mathcal{L}f(s) = \sum_{n=0}^{\infty} \frac{a_n}{s^{n+1}},$$

assuming $\Re(s) > -1$ and $|s| > 1$.

Step 5: Analytic Continuation

By observing that the Laurent series

$$\sum_{n=0}^{\infty} \frac{a_n}{s^{n+1}}$$

is holomorphic on the region $\{s \in C : |s| > 1\}$, and that this region together with the previous half-space $\{s : \Re(s) > -1\}$ cover the punctured complex plane, we can use analytic continuation to extend $\mathcal{L}f(s)$ analytically to the punctured complex plane $\{s \in C : s \neq -1\}$.

Step 6: Combining the Bounds

In the region $|s| > 1$, the Laurent series representation yields the bound

$$\mathcal{L}f(s) = O\left(\frac{1}{|s|-1}\right).$$

By combining this with the bound

$$\mathcal{L}f(s) = O\left(\frac{1}{\Re(s)+1}\right)$$

for $\Re(s) > -1$, we conclude the bound

$$\mathcal{L}f(s) = O\left(\frac{1}{|s+1|^2}\right)$$

in a sufficiently small neighborhood of $s = -1$.

Step 7: Double Pole at $s = -1$

This bound implies that $\mathcal{L}f(s)$ has at most a double pole at $s = -1$, giving a Laurent series representation of the form

$$\mathcal{L}f(s) = \frac{a}{(s+1)^2} + \frac{b}{s+1} + h(s),$$

where $h(s)$ is holomorphic near $s = -1$. Evaluating $\mathcal{L}f(s)$ at $s = -1 + \varepsilon$ and sending $\varepsilon \to 0$ shows that a must be zero, reducing the series to

$$\mathcal{L}f(s) = \frac{b}{s+1} + h(s).$$

Step 8: Using Liouville's Theorem

Combining this Laurent series with the bound $\mathcal{L}f(s) = O\left(\frac{1}{|s|-1}\right)$ for $|s| > 1$, we use Liouville's theorem to show that $h(s)$ must vanish, leaving

$$\mathcal{L}f(s) = \frac{b}{s+1}.$$

Step 9: Final Formula for a_n

Comparing this with the power series representation $\mathcal{L}f(s) = \sum_{n=0}^{\infty} \frac{a_n}{s^{n+1}}$, we find

$$\frac{b}{s+1} = \sum_{n=0}^{\infty} \frac{a_n}{s^{n+1}}.$$

Expanding $\frac{1}{s+1}$ as a geometric series, we get

$$\frac{1}{s+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{s^{n+1}},$$

so

$$\frac{b}{s+1} = b\sum_{n=0}^{\infty} \frac{(-1)^n}{s^{n+1}}.$$

Comparing coefficients, we conclude

$$a_n = b(-1)^n.$$

Thus, $a_n = C(-1)^n$ for some constant C, solving the problem.